

Complete families of solutions for the Dirac equation: an application of bicomplex pseudoanalytic function theory and transmutation operators

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Abstract

The Dirac equation with a scalar and an electromagnetic potentials is considered. In the time-harmonic case and when all the involved functions depend only on two spatial variables it reduces to a pair of decoupled bicomplex Vekua-type equations [8]. Using the technique developed for complex Vekua equations a system of exact solutions for the bicomplex equation is constructed under additional conditions, in particular when the electromagnetic potential is absent and the scalar potential is a function of one Cartesian variable. Introducing a transmutation operator relating the involved bicomplex Vekua equation with the Cauchy-Riemann equation we prove the expansion and the Runge approximation theorems corresponding to the constructed family of solutions.

1 Introduction

The Dirac system with a scalar and an electromagnetic potentials is considered. In [8] (see also [18]) it was shown that in the time-harmonic case and when the whole model is independent of one of the spatial variables the system reduces to a pair of decoupled Vekua-type equations which differ from the classical Vekua equations considered in the theory of generalized analytic or pseudoanalytic functions [2], [18], [28] by the fact that they are bicomplex. In [8] using this reduction as well as a procedure introduced by L. Bers, for an arbitrary scalar potential depending on one Cartesian variable an infinite family of solutions of the Dirac system was constructed. Nevertheless the completeness of this family in the linear space of all solutions was not proved due to the lack of some fundamental results in the theory of bicomplex Vekua equations such as the similarity principle and many other.

The constructed family of solutions is a system of formal powers generalizing those introduced by L. Bers onto the bicomplex situation. Meanwhile in the classical complex case there is a well developed theory of formal powers with the Runge-type approximation theorem and other related results (see [5] and references therein), in the bicomplex case up to now no such result was available even for simplest examples.

In the present work we prove the completeness of the family of solutions obtained in [8] by using so-called transmutation operators and some recent results on their mapping properties [6]. The notion and the name of the transmutation operator appeared in the work of J. Delsarte [11], [12] and

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later developed in [13], [23] and many other publications [1], [7], [22], [24], [25], [27]. Combining the results from [6] on mapping properties of the transmutation operators with the results from [21] on the construction of a transmutation operator for the Darboux transformed Schrödinger equation we obtain transmutation operators which relate the bicomplex Vekua equations arising from the Dirac system with the Cauchy-Riemann equation. Using this result we prove that the bicomplex pseudoanalytic formal powers are the result of application of a corresponding transmutation operator to the usual powers of the complex variable z . This together with the boundedness of the transmutation operator and of its inverse allows us to prove the expansion and the Runge approximation theorems for solutions of the considered bicomplex Vekua equations.

2 The Dirac system and bicomplex pseudoanalytic functions

Consider the Dirac operator with a scalar and an electromagnetic potentials

$$\mathbb{D} = \gamma_0 \partial_t + \sum_{k=1}^3 \gamma_k \partial_k + i \left(m + p_{el} \gamma_0 + \sum_{k=1}^3 A_k \gamma_k + p_{sc} \right)$$

where γ_j , $j = 0, 1, 2, 3$ are usual γ -matrices (see, e.g., [4], [26])

$$\begin{aligned} \gamma_0 &:= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma_1 &:= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma_2 &:= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &:= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$m \in \mathbb{R}$, p_{el} , A_k and p_{sc} are real valued functions.

We will denote the algebra of biquaternions or complex quaternions by $\mathbb{H}(\mathbb{C})$ with the standard basic quaternionic units denoted by $e_0 = 1$, e_1 , e_2 and e_3 . The complex imaginary unit is denoted by i as usual. The set of purely vectorial quaternions $q = \mathbf{q}$ is identified with the set of three-dimensional vectors.

The quaternionic conjugation of a biquaternion $q = q_0 + \mathbf{q}$ will be denoted as $\bar{q} = q_0 - \mathbf{q}$. Sometimes the following notation for the operator of multiplication from the right-hand side will be used $M^p q = q \cdot p$.

The main quaternionic differential operator introduced by Hamilton himself and sometimes called the Moisil-Theodoresco operator is defined on continuously differentiable biquaternion-valued functions of the real variables x_1 , x_2 and x_3 according to the rule

$$Dq = \sum_{k=1}^3 e_k \partial_k q,$$

where $\partial_k = \frac{\partial}{\partial x_k}$.

In [14] (see also [9], [15], [20]) a simple invertible matrix transformation was obtained which allows one to rewrite the classical Dirac equation in biquaternionic terms. Namely, the Dirac operator \mathbb{D} is equivalent to the biquaternionic operator

$$R = D - \partial_t M^{e_1} + \mathbf{a} + M^{-i(\tilde{p}_{el} e_1 - i(\tilde{p}_{sc} + m) e_2)}$$

where $\mathbf{a} = i(\tilde{A}_1 e_1 + \tilde{A}_2 e_2 - \tilde{A}_3 e_3)$ and the notation “ \sim ” means the reflection with respect to x_3 , $\tilde{f} := f(t, x_1, x_2, -x_3)$. Note that in the absence of the electromagnetic potential the operator R becomes real quaternionic which is an important property (see [19]).

In what follows we assume that potentials are time-independent and consider solutions with a fixed energy: $\Phi(t, \mathbf{x}) = \Phi_\omega(\mathbf{x})e^{i\omega t}$. The equation for Φ_ω has the form

$$\mathbb{D}_\omega \Phi_\omega = 0 \quad \text{in } \hat{G} \quad (1)$$

where \hat{G} is a domain in \mathbb{R}^3 ,

$$\mathbb{D}_\omega = i\omega\gamma_0 + \sum_{k=1}^3 \gamma_k \partial_k + i \left(m + p_{el}\gamma_0 + \sum_{k=1}^3 A_k \gamma_k + p_{sc} \right).$$

Under the mentioned above matrix transformation the operator \mathbb{D}_ω turns into its biquaternionic counterpart

$$R_\omega = D + \mathbf{a} + M^{\mathbf{b}}$$

with $\mathbf{b} = -i((\tilde{p}_{el} + \omega)e_1 - i(\tilde{p}_{sc} + m)e_2)$. Thus, equation (1) turns into the complex quaternionic equation

$$R_\omega q = 0 \quad (2)$$

where q is a complex quaternion valued function. In what follows we study this equation.

Let us introduce the following notation. For any biquaternion q we denote by Q_1 and Q_2 its bicomplex components:

$$Q_1 = q_0 + q_3 e_3 \quad \text{and} \quad Q_2 = q_2 - q_1 e_3.$$

Then q can be represented as follows $q = Q_1 + Q_2 e_2$. For the operator D we have $D = D_1 + D_2 e_2$ with $D_1 = e_3 \partial_3$ and $D_2 = \partial_2 - \partial_1 e_3$. Notice that $\mathbf{b} = B e_2$ with $B = -(\tilde{p}_{sc} + m) + i(\tilde{p}_{el} + \omega)e_3$, $\mathbf{a} = A_1 + A_2 e_2$ with $A_1 = a_3 e_3$ and $A_2 = a_2 - a_1 e_3$.

We obtain that equation (2) is equivalent to the system

$$D_1 Q_1 - D_2 \overline{Q}_2 + A_1 Q_1 - A_2 \overline{Q}_2 - \overline{B} Q_2 = 0, \quad (3)$$

$$D_2 \overline{Q}_1 + D_1 Q_2 + A_2 \overline{Q}_1 + A_1 Q_2 + B Q_1 = 0, \quad (4)$$

where Q_1 and Q_2 are bicomplex components of q . We stress that the system (3), (4) is equivalent to the Dirac equation in γ -matrices (1).

Let us suppose all fields in our model to be independent of x_3 , and $A_1 = a_3 e_3 \equiv 0$. Then the system (3), (4) decouples, and we obtain two separate bicomplex equations [8], [18]

$$\overline{D}_2 Q_2 = -\overline{A}_2 Q_2 - B \overline{Q}_2, \quad \text{and} \quad \overline{D}_2 Q_1 = -\overline{A}_2 Q_1 - \overline{B} \overline{Q}_1.$$

Denote $\overline{\partial} = \overline{D}_2$, $a = -\overline{A}_2$, $b = -B$, $w = Q_2$, $W = Q_1$, $z = x + y\mathbf{k}$, where $x = x_2$, $y = x_1$ and for convenience we denote $\mathbf{k} = e_3$. Then we reduce the Dirac equation with electromagnetic and scalar potentials independent of x_3 to a pair of Vekua-type equations

$$\overline{\partial} w = aw + b\overline{w} \quad (5)$$

and

$$\overline{\partial} W = aW + \overline{bW}. \quad (6)$$

3 Some definitions and results from bicomplex pseudoanalytic function theory

Definition 1 We consider \mathbb{B} -valued functions of two real variables x and y . Denote $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial y})$ and $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - \mathbf{k}\frac{\partial}{\partial y})$. An equation of the form

$$\bar{\partial}w = aw + b\bar{w}, \quad (7)$$

where w , a and b are \mathbb{B} -valued functions is called a bicomplex Vekua equation. When all the involved functions have their values in $\mathbb{C}_{\mathbf{k}}$ only, equation (7) becomes the well known complex Vekua equation (see [18], [28]). We will assume that $w \in C^1(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is an open domain and a, b are Hölder continuous in Ω .

When $a \equiv 0$ and $b = \frac{\bar{\partial}\phi}{\phi}$ where $\phi : \bar{\Omega} \rightarrow \mathbb{C}_i$ possesses Hölder continuous partial derivatives in Ω and $\phi(x, y) \neq 0, \forall (x, y) \in \bar{\Omega}$ we will say that the bicomplex Vekua equation

$$\bar{\partial}w = \frac{\bar{\partial}\phi}{\phi}\bar{w} \quad (8)$$

is a Vekua equation of the main type or the main Vekua equation.

For classical complex Vekua equations Bers introduced [2] the notions of a generating pair, generating sequence, formal powers and Taylor series in formal powers. As was shown in [8], [18] the definition of these notions can be extended onto the bicomplex situation. Here we briefly recall the main definitions.

Definition 2 A pair of \mathbb{B} -valued functions F and G possessing Hölder continuous partial derivatives in Ω with respect to the real variables x and y is said to be a generating pair if it satisfies the inequality

$$\text{Vec}(\bar{F}G) \neq 0 \quad \text{in } \Omega. \quad (9)$$

Condition (9) implies that every bicomplex function w defined in a subdomain of Ω admits the unique representation $w = \phi F + \psi G$ where the functions ϕ and ψ are scalar (\mathbb{C}_i -valued).

Remark 3 When $F \equiv 1$ and $G \equiv \mathbf{k}$ the corresponding bicomplex Vekua equation is

$$\bar{\partial}w = 0, \quad (10)$$

and its study in fact reduces to the complex analytic function theory [6]. Indeed, consider the following pair of idempotents $\mathbf{P}^+ = \frac{1}{2}(1 + i\mathbf{k})$ and $\mathbf{P}^- = \frac{1}{2}(1 - i\mathbf{k})$ ($(\mathbf{P}^\pm)^2 = \mathbf{P}^\pm$). Then the functions \mathbf{P}^+w and \mathbf{P}^-w are necessarily antiholomorphic and holomorphic respectively. Indeed, application of \mathbf{P}^+ and \mathbf{P}^- to (10) gives us

$$\partial_z \mathbf{P}^+w = 0 \quad \text{and} \quad \partial_{\bar{z}} \mathbf{P}^-w = 0 \quad (11)$$

where $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Moreover, $\mathbf{P}^+w = \mathbf{P}^+(u + jv) = \mathbf{P}^+(u - iv)$ and $\mathbf{P}^-w = \mathbf{P}^-(u + iv)$. Due to (11) the scalar functions $w^+ := u - iv$ and $w^- := u + iv$ are antiholomorphic and holomorphic respectively. We stress that w^+ is not necessarily a complex conjugate of w^- (u and v are \mathbb{C}_i -valued).

Let us notice that due to the equivalence of (10) and (11) we have that a bicomplex solution w of (10) admits a convergent Taylor series $w(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if the series $\sum_{n=0}^{\infty} a_n^+ (z^+)^n$ and $\sum_{n=0}^{\infty} a_n^- (z^-)^n$ corresponding to w^+ and w^- respectively converge (here a_n^\pm and z^\pm are scalars, $a_n^\pm = \text{Sc}(a_n) \mp i \text{Vec}(a_n)$ and $z^\pm = x \mp iy$). In particular, the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ has the form $R = \min\{R_+, R_-\}$ where $1/R_\pm = \limsup_{n \rightarrow \infty} |a_n^\pm|^{1/n}$.

Assume that (F, G) is a generating pair in a domain Ω .

Definition 4 Let the \mathbb{B} -valued function w be defined in a neighborhood of $z_0 \in \Omega \subset \mathbb{C}_{\mathbf{k}}$. In a complete analogy with the complex case we say that at z_0 the function w possesses the (F, G) -derivative $\dot{w}(z_0)$ if the (finite) limit

$$\dot{w}(z_0) = \lim_{z \rightarrow z_0} \frac{w(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0} \quad (12)$$

exists where λ_0 and μ_0 are the unique scalar constants such that $w(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0)$.

Similarly to the complex case (see, e.g., [18, Chapter 2]) it is easy to show that if $\dot{w}(z_0)$ exists then at z_0 , $\bar{\partial}w$ and ∂w exist and equations

$$\bar{\partial}w = a_{(F,G)}w + b_{(F,G)}\bar{w} \quad (13)$$

and

$$\dot{w} = \partial w - A_{(F,G)}w - B_{(F,G)}\bar{w} \quad (14)$$

hold, where $a_{(F,G)}$, $b_{(F,G)}$, $A_{(F,G)}$ and $B_{(F,G)}$ are the *characteristic coefficients* of the pair (F, G) defined by the formulas

$$\begin{aligned} a_{(F,G)} &= -\frac{\bar{F}\bar{\partial}G - \bar{G}\bar{\partial}F}{F\bar{G} - \bar{F}G}, & b_{(F,G)} &= \frac{F\bar{\partial}G - G\bar{\partial}F}{F\bar{G} - \bar{F}G}, \\ A_{(F,G)} &= -\frac{\bar{F}\partial G - \bar{G}\partial F}{F\bar{G} - \bar{F}G}, & B_{(F,G)} &= \frac{F\partial G - G\partial F}{F\bar{G} - \bar{F}G}. \end{aligned}$$

Note that $F\bar{G} - \bar{F}G = -2\mathbf{k} \text{Vec}(\bar{F}G) \neq 0$.

If $\bar{\partial}w$ and ∂w exist and are continuous in some neighborhood of z_0 , and if (13) holds at z_0 , then $\dot{w}(z_0)$ exists, and (14) holds. Let us notice that F and G possess (F, G) -derivatives, $\dot{F} \equiv \dot{G} \equiv 0$ and the following equalities are valid which determine the characteristic coefficients uniquely

$$\begin{aligned} \bar{\partial}F &= a_{(F,G)}F + b_{(F,G)}\bar{F}, & \bar{\partial}G &= a_{(F,G)}G + b_{(F,G)}\bar{G}, \\ \partial F &= A_{(F,G)}F + B_{(F,G)}\bar{F}, & \partial G &= A_{(F,G)}G + B_{(F,G)}\bar{G}. \end{aligned}$$

If the (F, G) -derivative of a \mathbb{B} -valued function $w = \phi F + \psi G$ (where the functions ϕ and ψ are scalar) exists, besides the form (14) it can also be written as follows $\dot{w} = \partial\phi F + \partial\psi G$.

Definition 5 Let (F, G) and (F_1, G_1) – be two generating pairs in Ω . (F_1, G_1) is called *successor* of (F, G) and (F, G) is called *predecessor* of (F_1, G_1) if

$$a_{(F_1, G_1)} = a_{(F, G)} \quad \text{and} \quad b_{(F_1, G_1)} = -B_{(F, G)}.$$

By analogy with the complex case we have the following statement.

Theorem 6 Let w be a bicomplex (F, G) -pseudoanalytic function and let (F_1, G_1) be a successor of (F, G) . Then w is a bicomplex (F_1, G_1) -pseudoanalytic function.

Definition 7 Let (F, G) be a generating pair. Its *adjoint generating pair* $(F, G)^* = (F^*, G^*)$ is defined by the formulas

$$F^* = -\frac{2\bar{F}}{F\bar{G} - \bar{F}G}, \quad G^* = \frac{2\bar{G}}{F\bar{G} - \bar{F}G}.$$

The (F, G) -integral is defined as follows

$$\int_{\Gamma} W d_{(F,G)} z = \frac{1}{2} \left(F(z_1) \operatorname{Sc} \int_{\Gamma} G^* W dz + G(z_1) \operatorname{Sc} \int_{\Gamma} F^* W dz \right)$$

where Γ is a rectifiable curve leading from z_0 to z_1 .

If $W = \phi F + \psi G$ is a bicomplex (F, G) -pseudoanalytic function where ϕ and ψ are complex valued functions then

$$\int_{z_0}^z W d_{(F,G)} z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z), \quad (15)$$

and this integral is path-independent and represents the (F, G) -antiderivative of W .

Definition 8 A sequence of generating pairs $\{(F_m, G_m)\}$, $m = 0, \pm 1, \pm 2, \dots$, is called a generating sequence if (F_{m+1}, G_{m+1}) is a successor of (F_m, G_m) . If $(F_0, G_0) = (F, G)$, we say that (F, G) is embedded in $\{(F_m, G_m)\}$.

Let W be a bicomplex (F, G) -pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of W by the recursion formula

$$W^{[0]} = W; \quad W^{[m+1]} = \frac{d_{(F_m, G_m)} W^{[m]}}{dz}, \quad m = 1, 2, \dots$$

Definition 9 The formal power $Z_m^{(0)}(a, z_0; z)$ with center at $z_0 \in \Omega$, coefficient a and exponent 0 is defined as the linear combination of the generators F_m, G_m with scalar constant coefficients λ, μ chosen so that $\lambda F_m(z_0) + \mu G_m(z_0) = a$. The formal powers with exponents $n = 0, 1, 2, \dots$ are defined by the recursion formula

$$Z_m^{(n+1)}(a, z_0; z) = (n+1) \int_{z_0}^z Z_{m+1}^{(n)}(a, z_0; \zeta) d_{(F_m, G_m)} \zeta. \quad (16)$$

This definition implies the following properties.

1. $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m) -pseudoanalytic function of z .
2. If a' and a'' are scalar constants, then

$$Z_m^{(n)}(a' + \mathbf{k}a'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(\mathbf{k}, z_0; z).$$

3. The formal powers satisfy the differential relations

$$\frac{d_{(F_m, G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

4. The asymptotic formulas

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \rightarrow z_0$$

hold.

The case of the main bicomplex Vekua equation is of a special interest also due to the following relation with the stationary Schrödinger equation.

Theorem 10 [16] *Let $W = W_1 + \mathbf{k}W_2$ be a solution of the main bicomplex Vekua equation*

$$\bar{\partial}W = \frac{\bar{\partial}\phi}{\phi}\bar{W} \quad \text{in } \Omega \quad (17)$$

where $W_1 = \text{Sc } W$, $W_2 = \text{Vec } W$ and the \mathbb{C}_i -valued function ϕ is a nonvanishing solution of the equation

$$-\Delta u + q_1(x, y)u = 0 \quad \text{in } \Omega \quad (18)$$

where q_1 is a continuous \mathbb{C}_i -valued function. Then W_1 is a solution of (18) in Ω and W_2 is a solution of the associated Schrödinger equation

$$-\Delta v + q_2(x, y)v = 0 \quad \text{in } \Omega \quad (19)$$

where $q_2 = 8\frac{\bar{\partial}\phi\partial\phi}{\phi^2} - q_1$.

We need the following notation. Let w be a \mathbb{B} -valued function defined on a simply connected domain Ω with $w_1 = \text{Sc } w$ and $w_2 = \text{Vec } w$ such that

$$\frac{\partial w_1}{\partial y} - \frac{\partial w_2}{\partial x} = 0, \quad \forall (x, y) \in \Omega, \quad (20)$$

and let $\Gamma \subset \Omega$ be a rectifiable curve leading from (x_0, y_0) to (x, y) . Then the integral

$$\bar{A}w(x, y) := 2 \left(\int_{\Gamma} w_1 dx + w_2 dy \right)$$

is path-independent, and all \mathbb{C}_i -valued solutions φ of the equation $\bar{\partial}\varphi = w$ in Ω have the form $\varphi(x, y) = \bar{A}w(x, y) + c$ where c is an arbitrary \mathbb{C}_i -constant. In other words the operator \bar{A} denotes the well known operation for reconstructing the potential function from its gradient.

Theorem 11 [16] *Let W_1 be a \mathbb{C}_i -valued solution of the Schrödinger equation (18) in a simply connected domain Ω . Then a \mathbb{C}_i -valued solution W_2 of the associated Schrödinger equation (19) such that $W_1 + \mathbf{k}W_2$ is a solution of (17) in Ω can be constructed according to the formula*

$$W_2 = \frac{1}{\phi}\bar{A}\left(\mathbf{k}\phi^2\bar{\partial}\left(\frac{W_1}{\phi}\right)\right) + \frac{c_1}{\phi}$$

where c_1 is an arbitrary \mathbb{C}_i -constant.

Vice versa, given a solution W_2 of (19), the corresponding solution W_1 of (18) such that $W_1 + \mathbf{k}W_2$ is a solution of (17) has the form

$$W_1 = -\phi\bar{A}\left(\frac{\mathbf{k}}{\phi^2}\bar{\partial}(\phi W_2)\right) + c_2\phi$$

where c_2 is an arbitrary \mathbb{C}_i -constant.

As was shown in [17] (see also [18]) a generating sequence can be obtained in a closed form, for example, in the case when ϕ has a separable form $\phi = S(s)T(t)$ where s and t are conjugate harmonic functions and S, T are arbitrary twice continuously differentiable functions. In practical terms this means that whenever the Schrödinger equation (18) admits a particular nonvanishing solution having the form $\phi = f(\xi)g(\eta)$ where (ξ, η) is one of the encountered in physics orthogonal coordinate systems in the plane a generating sequence corresponding to (17) can be obtained explicitly [18, Sect. 4.8]. The knowledge of a generating sequence allows one to construct the formal powers following Definition 9. This construction is a simple algorithm which can be quite easily and efficiently realized numerically

[5], [10]. Moreover, in the case of a complex main Vekua equation which in the notations admitted in the present paper corresponds to the case of ϕ being a real-valued function (then the main bicomplex Vekua equation decouples into two main complex Vekua equations) the completeness of the system of formal powers was proved [5] in the sense that any pseudoanalytic in Ω and Hölder continuous on $\partial\Omega$ function can be approximated uniformly and arbitrarily closely by a finite linear combination of the formal powers. The real parts of the complex pseudoanalytic formal powers represent then a complete system of solutions of one Schrödinger equation meanwhile the imaginary parts give us a complete system of solutions of the associated Schrödinger equation.

4 Transmutation operators and a complete family of solutions of the Dirac equation

Let us consider the following situation $p_{sc} = p(x)$, $p_{el} = 0$ and $\vec{A} = 0$. Then the Dirac equation is equivalent to the pair of bicomplex Vekua equations

$$\bar{\partial}w = b\bar{w} \quad (21)$$

$$\bar{\partial}W = \bar{b}\bar{W} \quad (22)$$

where $b = p(x) + m - i\omega\mathbf{k}$.

Let P denote an antiderivative of p . Consider the function

$$\phi(x, y) = e^{P(x)+mx+i\omega y}$$

Note that $\bar{\partial}\phi/\phi = \bar{b}$. Then if W is a solution of (22) then the complex valued function $W_1 = Sc(W)$ is a solution of the Schrödinger equation

$$(-\Delta + \nu)W_1 = 0, \text{ with } \nu(x) = p'(x) + (p(x) + m)^2 - \omega^2$$

and the complex valued function $W_2 = Vec(W)$ is a solution of the associated Schrödinger equation

$$(-\Delta + \mu)W_2 = 0, \text{ with } \mu(x) = -p'(x) + (p(x) + m)^2 - \omega^2$$

On the other hand equation (22) can be written as a main Vekua equation

$$\left(\bar{\partial} - \frac{\bar{\partial}\phi}{\phi}C\right)W = 0 \quad (23)$$

where

$$\phi(x, y) = f(x)g(y) \text{ with } f(x) = e^{P(x)+mx} \text{ and } g(y) = e^{i\omega y}$$

Notice that f and g are complex valued functions. We assume that their domains of definitions are finite segments $[-a, a]$ and $[-b, b]$ respectively. Assuming that $p \in C^1[-a, a]$ we obtain that f and g are nonvanishing C^2 -functions. The separable form of ϕ allows us to write down a generating pair associated with equation (23) $(F, G) = (\phi, \mathbf{k}/\phi)$ as well as a generating sequence of the period two embedding this generating pair

$$\begin{aligned} (F, G) &= (\phi, \mathbf{k}/\phi); & (F_1, G_1) &= (\phi/f^2, \mathbf{k}f^2/\phi) \\ (F_2, G_2) &= (F, G) & ; & (F_3, G_3) = (F_1, G_1) & ; \dots \end{aligned}$$

The corresponding formal powers can be constructed as follows. We consider the formal powers with the centre in the origin and for simplicity assume that $f(0) = 1$ (for g this is also the case). Define the following systems of functions $\{\varphi_k\}_{k=0}^\infty$ and $\{\psi_k\}_{k=0}^\infty$

$$\varphi_k(x) = \begin{cases} f(x)X^{(k)}(x), & k \text{ odd} \\ f(x)\tilde{X}^{(k)}(x), & k \text{ even} \end{cases} \quad (24)$$

where

$$\begin{aligned} X^{(0)}(x) &= \tilde{X}^{(0)}(x) = 1 \\ X^{(n)}(x) &= n \int_o^x X^{(n-1)}(\rho) [f^2(\rho)]^{(-1)^n} d\rho \\ \tilde{X}^{(n)}(x) &= n \int_o^x \tilde{X}^{(n-1)}(\rho) [f^2(\rho)]^{(-1)^{n-1}} d\rho \end{aligned}$$

and

$$\psi_k(y) = \begin{cases} g(y)Y^{(k)}(y), & k \text{ odd} \\ g(y)\tilde{Y}^{(k)}(y), & k \text{ even} \end{cases} \quad (25)$$

where

$$\begin{aligned} Y^{(0)}(y) &= \tilde{Y}^{(0)}(y) = 1 \\ Y^{(n)}(y) &= n \int_o^y Y^{(n-1)}(\xi) [g^2(\xi)]^{(-1)^n} d\xi \\ \tilde{Y}^{(n)}(y) &= n \int_o^y \tilde{Y}^{(n-1)}(\xi) [g^2(\xi)]^{(-1)^{n-1}} d\xi. \end{aligned}$$

Then the formal powers based on the given generating sequence are defined by the formulas

$$Z^{(n)}(\alpha, 0; z) = \phi(x, y) Sc_* Z^{(n)}(\alpha, 0; z) + \frac{\mathbf{k}}{\phi(x, y)} Vec_* Z^{(n)}(\alpha, 0; z) \quad (26)$$

where

$$*_Z^{(n)}(\alpha, 0; z) = \begin{cases} \alpha' \sum_{m=0}^n \binom{n}{m} X^{(n-m)} \mathbf{k}^m \tilde{Y}^{(m)} + \mathbf{k} \alpha' \sum_{m=0}^n \binom{n}{m} \tilde{X}^{(n-m)} \mathbf{k}^m Y^{(m)}, & n \text{ odd} \\ \alpha' \sum_{m=0}^n \binom{n}{m} \tilde{X}^{(n-m)} \mathbf{k}^m \tilde{Y}^{(m)} + \mathbf{k} \alpha' \sum_{m=0}^n \binom{n}{m} X^{(n-m)} \mathbf{k}^m Y^{(m)}, & n \text{ even.} \end{cases} \quad (27)$$

In a similar way the formal powers corresponding to (21) can be constructed, they will be denoted as $\tilde{Z}^{(n)}(\alpha, 0; z)$. Notice that a generating pair for (21) is given by

$$\tilde{F}_0 = \frac{g}{f} \quad \text{and} \quad \tilde{G}_0 = \mathbf{k} \frac{f}{g}.$$

In [6] it was shown that for functions f and g satisfying the above conditions there exist the transmutation operators T_f and T_g defined as follows

$$T_f[u(x)] = u(x) + \int_{-x}^x \mathbf{K}(x, t; f'(0)) \cdot u(t) dt \quad (28)$$

where the kernel $\mathbf{K}(x, t; f'(0))$ is given by

$$\mathbf{K}(x, t; f'(0)) = \frac{f'(0)}{2} + K(x, t) + \frac{f'(0)}{2} \int_t^x [K(x, s) - K(x, -s)] ds$$

and the function $K(x, t)$ is the unique solution of the Goursat problem (see [24])

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} - q(x) \right) K(x, t) = \frac{\partial^2}{\partial t^2} K(x, t) \\ K(x, x) = \frac{1}{2} \int_0^x q(s) ds; \quad K(x, -x) = 0 \end{cases}$$

and

$$T_g[v(y)] = v(y) + \int_{-y}^y \tilde{\mathbf{K}}(y, t; g'(0))v(t)dt$$

where

$$\tilde{\mathbf{K}}(y, t; g'(0)) = \frac{g'(0)}{2} + \tilde{K}(y, t) + \frac{g'(0)}{2} \int_t^y [\tilde{K}(y, s) - \tilde{K}(y, -s)] ds$$

and the function $\tilde{K}(x, t)$ is the unique solution of the Goursat problem

$$\begin{cases} \left(\frac{\partial^2}{\partial y^2} - \tilde{q}(y) \right) \tilde{K}(y, t) = \frac{\partial^2}{\partial t^2} \tilde{K}(y, t) \\ \tilde{K}(y, y) = \frac{1}{2} \int_0^y \tilde{q}(s) ds; \quad \tilde{K}(y, -y) = 0 \end{cases}$$

with $q = f''/f$ and $\tilde{q} = g''/g$.

Moreover, T_f and T_g satisfy the relations

$$T_f[x^k] = \varphi_k \quad \text{and} \quad T_g[y^k] = \psi_k, \quad \forall k \in \mathbb{N}_0. \quad (29)$$

We will need similar systems of functions $\{\tilde{\varphi}_k\}_{k=0}^\infty$ and $\{\tilde{\psi}_k\}_{k=0}^\infty$ corresponding to $1/f$ and $1/g$ respectively,

$$\tilde{\varphi}_k(x) = \begin{cases} \frac{1}{f(x)} X^{(k)}(x), & k \text{ even} \\ \frac{1}{f(x)} \tilde{X}^{(k)}(x), & k \text{ odd} \end{cases} \quad (30)$$

$$\tilde{\psi}_k(y) = \begin{cases} \frac{1}{g(y)} Y^{(k)}(y), & k \text{ even} \\ \frac{1}{g(y)} \tilde{Y}^{(k)}(y), & k \text{ odd.} \end{cases} \quad (31)$$

For these systems of functions another pair of transmutations $T_{1/f}$ and $T_{1/g}$ is constructed (see [21]), one of the representations of which can be given by the equalities

$$T_{1/f}u(x) = \frac{1}{f(x)} \left\{ \int_0^x f(\eta) T_f[\partial u(\eta)] d\eta + u(0) \right\},$$

$$T_{1/g}v(y) = \frac{1}{g(y)} \left\{ \int_0^y g(\eta) T_g[\partial v(\eta)] d\eta + v(0) \right\}.$$

They satisfy the equalities

$$T_{1/f}[x^k] = \tilde{\varphi}_k \quad \text{and} \quad T_{1/g}[y^k] = \tilde{\psi}_k, \quad \forall k \in \mathbb{N}_0. \quad (32)$$

The operators $T_{1/f}$ and $T_{1/g}$ admit the representations as Volterra integral operators [21],

$$T_{1/f}u(x) = u(x) + \int_{-x}^x \mathbf{K}_2(x, t; -f'(0))u(t) dt,$$

where the kernel $\mathbf{K}_2(x, t; -f'(0))$ has the form

$$\mathbf{K}_2(x, t; -f'(0)) = -\frac{1}{f(x)} \left(\int_{-t}^x \partial_t \mathbf{K}_1(s, t; f'(0))f(s) ds + \frac{f'(0)}{2} f(-t) \right)$$

and the formulas for $T_{1/g}$ are completely analogous with an obvious substitution of f by g .

The introduced transmutation operators satisfy interesting commutation equalities.

Corollary 12 [21] *The following operator equalities hold on C^1 -functions of the respective variables*

$$\partial_x f T_{1/f} = f T_f \partial_x, \quad \partial_x \frac{1}{f} T_f = \frac{1}{f} T_{1/f} \partial_x. \quad (33)$$

$$\partial_y g T_{1/g} = g T_g \partial_y, \quad \partial_y \frac{1}{g} T_g = \frac{1}{g} T_{1/g} \partial_y. \quad (34)$$

Consider the operators projecting onto the scalar and the vector parts respectively

$$P^+ = \frac{1}{2} (I + C) \quad \text{and} \quad P^- = \frac{1}{2\mathbf{k}} (I - C).$$

Let us introduce the following operators

$$\mathbf{T}_0 = T_f T_g P^+ + \mathbf{k} T_{1/f} T_{1/g} P^- \quad (35)$$

and

$$\mathbf{T}_1 = T_{1/f} T_g P^+ + \mathbf{k} T_f T_{1/g} P^-. \quad (36)$$

From now on let $\Omega \subset \overline{R} = [-a, a] \times [-b, b]$ be a simply connected domain such that together with any point (x, y) belonging to Ω the rectangle with the vertices (x, y) , $(-x, y)$, $(x, -y)$ and $(-x, -y)$ also belongs to Ω . In such a domain application of operators \mathbf{T}_0 and \mathbf{T}_1 is meaningful.

Proposition 13 *The following equalities hold for any \mathbb{B} -valued, continuously differentiable function w defined in Ω .*

$$\left(\bar{\partial} - \frac{\bar{\partial}\phi}{\phi} C \right) \mathbf{T}_0 w = \mathbf{T}_1 (\bar{\partial} w), \quad \left(\bar{\partial} + \frac{\partial\phi}{\phi} C \right) \mathbf{T}_1 w = \mathbf{T}_0 (\bar{\partial} w). \quad (37)$$

$$\left(\partial - \frac{\partial\phi}{\phi} C \right) \mathbf{T}_0 w = \mathbf{T}_1 (\partial w), \quad \left(\partial + \frac{\bar{\partial}\phi}{\phi} C \right) \mathbf{T}_1 w = \mathbf{T}_0 (\partial w). \quad (38)$$

Proof. The proof consists in a direct calculation with the aid of the relations from Corollary 12. ■

An immediate corollary of equalities (37) is the fact that the operator \mathbf{T}_0 maps bicomplex analytic functions into $(\phi, \mathbf{k}/\phi)$ -pseudoanalytic, i.e., into solutions of (23) and the operator \mathbf{T}_1 maps bicomplex analytic functions into $(\frac{g}{f}, \mathbf{k}\frac{f}{g})$ -pseudoanalytic i.e., into solutions of the equation

$$\left(\bar{\partial} + \frac{\partial\phi}{\phi} C \right) W = 0. \quad (39)$$

Moreover, they map powers of the variable z into corresponding formal powers.

Proposition 14 *For any $z \in \Omega$ and $a \in \mathbb{B}$ the following equalities are valid*

$$\mathbf{T}_0[az^n] = Z^{(n)}(a, 0; z) \quad \text{and} \quad \mathbf{T}_1[az^n] = Z_1^{(n)}(a, 0; z).$$

Proof. The proof consists in the observation that for $a = a' + \mathbf{k}b'$ and $z = x + \mathbf{k}y$ one has

$$az^n = (a' + \mathbf{k}b') \sum_{m=0}^n \binom{n}{m} x^{n-m} \mathbf{k}^m y^m$$

and the result follows from the formulas (26), (27) by application of the mapping properties (29), (32). ■

Notice that both \mathbf{T}_0 and \mathbf{T}_1 are bounded operators on the space of continuous functions with respect to the norm $\|w\| = \max(|u| + |v|)$ where $w = u + \mathbf{k}v$. Indeed, consider $\|\mathbf{T}_0 w\| = \max(|T_f T_g u| +$

$|T_{1/f}T_{1/g}v| \leq M_1 \max|u| + M_2 \max|v|$ where the constants M_1 and M_2 depend only on the corresponding kernels of the bounded Volterra operators T_f , T_g , $T_{1/f}$ and $T_{1/g}$. Then $\|\mathbf{T}_0 w\| \leq M \|w\|$ where $M = \max\{M_1, M_2\}$. The proof for the operator \mathbf{T}_1 is analogous. Moreover, \mathbf{T}_0^{-1} and \mathbf{T}_1^{-1} are bounded as well (the form of the inverses for T_f , T_g , $T_{1/f}$ and $T_{1/g}$ can be found in [21]) due to the fact that their integral kernels enjoy the same regularity properties as the kernels of \mathbf{T}_0 and \mathbf{T}_1 .

Let us establish another useful fact concerning the mapping properties of the operators \mathbf{T}_0 and \mathbf{T}_1 .

Proposition 15 *Let w be a bicomplex analytic function in Ω and $W = \mathbf{T}_0 w$ be a corresponding solution of (23). Then*

$$\mathbf{T}_0 \left(\partial^{(2n)} w \right) = W^{[2n]} \quad \text{and} \quad \mathbf{T}_1 \left(\partial^{(2n-1)} w \right) = W^{[2n-1]}, \quad n = 1, 2, \dots \quad (40)$$

Proof. From (38) we have

$$\dot{W} = \mathbf{T}_1 (\partial w). \quad (41)$$

\dot{W} is a solution of the succeeding Vekua equation (39). Denote $W_1 = \dot{W}$. Any solution of (39) is the image of a bicomplex analytic function under the action of the operator \mathbf{T}_1 , so $W_1 = \mathbf{T}_1 w_1$. Due to (38) we have $\dot{W}_1 = \mathbf{T}_0 (\partial w_1)$. Thus, $\ddot{W} = \mathbf{T}_0 (\partial^2 w)$ because from (41) $w_1 = \partial w$. Now (40) can be easily proved by induction. ■

The established relations from Propositions 13 and 14 together with the fact that \mathbf{T}_0 and \mathbf{T}_1 are bounded operators together with their respective inverses allow us to transfer several results from analytic function theory onto the solutions of the bicomplex Vekua equations under consideration and as hence onto the solutions of the Dirac system with a scalar potential being a function of one Cartesian variable. Here we give two examples of such results.

Theorem 16 *Let W be a solution of (23) in a disk D with the center in the origin and radius R . Then it can be expanded into a Taylor series in formal powers*

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, 0; z)$$

with the radius of convergence R . The series converges normally in D and the coefficients a_n have the form

$$a_n = \frac{W^{[n]}(0)}{n!}.$$

Proof. Consider $w = \mathbf{T}_0^{-1} W$. It is a bicomplex analytic function, so we have that it can be expanded into a Taylor series $w(z) = \sum_{n=0}^{\infty} a_n z^n$ with the coefficients $a_n = \frac{d^n w(0)}{dz^n} / n!$. Application of \mathbf{T}_0 gives us

a series for W , $W(z) = \mathbf{T}_0 w(z) = \sum_{n=0}^{\infty} \mathbf{T}_0 [a_n z^n] = \sum_{n=0}^{\infty} Z^{(n)}(a_n, 0; z)$. Due to the uniform boundedness of \mathbf{T}_0 the radius of convergence of the series is preserved. Note that the Taylor coefficients coincide. In order to finish the proof we use Proposition 15 and the fact that both operators \mathbf{T}_0 and \mathbf{T}_1 preserve the values of a function in the origin. This is obvious from their definition and from the Volterra integral form of the operators T_f , T_g , $T_{1/f}$ and $T_{1/g}$ (see, e.g., (28)). Thus, $W^{[2n]}(0) = \mathbf{T}_0 (\partial^{(2n)} w)(0) = \partial^{(2n)} w(0)$ and $W^{[2n-1]}(0) = \mathbf{T}_1 (\partial^{(2n-1)} w)(0) = \partial^{(2n-1)} w(0)$, $n = 1, 2, \dots$ ■

Theorem 17 *Any solution W of (23) in Ω can be approximated arbitrarily closely on any compact subset K of Ω by a finite combination of formal powers (a formal polynomial) $\sum_{n=0}^N Z^{(n)}(a_n, 0; z)$.*

Proof. Consider $w = \mathbf{T}_0^{-1}W$. Due to the Runge approximation theorem the function w can be arbitrarily closely approximated by a polynomial in z . Then due to the boundedness of \mathbf{T}_0 and \mathbf{T}_0^{-1} and Proposition 14 we obtain the required result for W . ■

This theorem in fact means the completeness of the family of functions

$$\left\{ Z^{(n)}(1, 0; z), \quad Z^{(n)}(\mathbf{k}, 0; z) \right\}_{n=0}^{\infty}$$

in the space of all solutions of the Vekua equation (23). A similar fact is true for equation (39) and corresponding formal powers. The combination of both families of formal powers gives us a complete family of solutions of the Dirac equation (2) in the considered case.

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